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Derivation of Closing Speed as a Function of Dissipated Energy

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ABSTRACT

In accident reconstruction, a relationship frequently used for completely inelastic collisions is one between relative speed (or closing speed), vehicle weights and total energy dissipated in deforming the vehicles during collision (crush energy). Derivation of this equation is presented in various accident reconstruction literature, but always for collinear collisions, i.e. vehicles traveling along the same straight line. A derivation is presented here for the general case, which shows the same relationship is valid for any angle of approach between the vehicles, not just 0^{o} or 180^{o}. A supplemental derivation is presented which yields the relationship for all collisions, not just completely inelastic collisions.

INTRODUCTION

In a large fraction of vehicular accident reconstructions, the items of interest are the pre-impact speeds of the vehicles. In many cases, knowledge of the closing speed, or relative speed, of the vehicles at impact can determine directly the speed of one of the vehicles within reasonably narrow limits. The closing speed can be determined from measured damage or "crush" of the vehicles, vehicle weights and stiffness coefficients of the vehicles obtained from crash tests.[1]

For example, if the struck vehicle is known to be at rest at impact, the striking vehicle's speed can be determined directly since its speed is the calculated closing speed. Also, if one vehicle is known to be moving slowly, e.g. just started from rest at a traffic light or stop sign, and is struck perpendicularly by a somewhat faster moving vehicle the difference between the closing speed and the faster vehicle's speed will be small. Consider a vehicle known to have started from rest and to have moved into an intersection such that its upper limit of speed is 10 mph. If it is struck by a vehicle going 40 mph at impact, the closing speed of 41.2 mph is only 3% higher than the striking vehicle's speed at impact.

In general, determination of pre-impact speeds can be achieved from the Law of Conservation of Momentum. This necessitates knowing the post-impact momentum of each vehicle and their pre-impact directions of travel. Measurements to determine post-impact momentum are frequently flawed by lack of road-surface marks to precisely indicate immediate post-impact velocity directions. Steer input can result from vehicle damage (damaged wheels) or inhomogeneities along the surface traveled after impact, thus point-of-impact to point-of-rest can be very different from the initial post-impact velocity direction. Also, the post impact initial speed can be uncertain due to unknown post-impact degree of braking (if any), uncertain degree of roll resistance due to damaged wheels or uncertain angle between post-impact directions of travel and the roll directions of the vehicle's wheels as well as various other sources of uncertainty. In these cases, closing speed at impact, as given in [1] and [2], and in equations (12) and (13) shown below, is an additional piece of information for solving for speeds and is a check for closing speed derived by other means.

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[1] See [1], p. 70-20 Exhibit 24, equations for crush energy for two, four and six equidistant crush measurements (one, three and five equal width regions of damage). Similar equations are also valid for odd numbers of equidistant crush measurements (even number of equal width regions of damage) as well as for more than six equidistant crush measurements. In general:

$$E = \frac{W}{(n-1)}\left[\begin{array}{l}(n-1)G + \\ \frac{A}{2}\left(C_1 + 2C_2 + - - - + 2C_{n-1} + C_n\right) \\ + \frac{B}{6}\left(\begin{array}{l}C_1^2 + 2C_2^2 + - - - + 2C_{n-1}^2 + C_n^2 \\ + C_1C_2 + C_2C_3 + - - - + C_{n-1}C_n\end{array}\right)\end{array}\right]\left(1 + Tan^2\Theta\right)$$

where n is the number of equidistant crush measurement, n-1 the number of equal width regions of damage between the measured crush depths, C_n , W the total width of the damaged area, A and B are experimentally determined stiffness coefficients, $G = A^2/2B$, E the energy dissipated in crushing the vehicle and Θ , the angle between PDOF and the normal to the struck surface.

The equation for closing speed from damage presented in the literature is usually derived for the special case of same direction in-line completely inelastic collisions (also called plastic collisions). Many collisions, e.g. intersection collisions, occur with the direction of travel of the two vehicles at impact forming some angle other than 0^o or 180^o. Furthermore, low speed collisions are not well approximated by completely inelastic collisions as they have coefficients of restitution significantly above zero, yet significantly less than one, and thus are not very well approximated by perfectly elastic collisions either.

Let us first examine the derivation of the closure speed for a completely inelastic non-collinear collision and then extend this to include all real world cases for $0 \le e < 1$, where e is the coefficient of restitution (including completely inelastic collisions, $e = 0$, and up to but not including perfectly elastic collisions, $e < 1$).

NON-COLLINEAR COMPLETELY INELASTIC COLLISION

The notation used is as follows: The magnitude of a vector, $\vec{X}$ is represented by $\left|\vec{X}\right| = X$, a scalar quantity. Velocity of A relative to E, $\vec{V}_{AE}$, has a magnitude $\left|\vec{V}_{AE}\right|$ or V_{AE}, which is a scalar quantity, the speed. The square of a vector is defined as $\vec{V}_{AB}^{\,2} = \vec{V}_{AB} \cdot \vec{V}_{AB} = \left|\vec{V}_{AB}\right|^2 = V_{AB}^2$, also a scalar quantity.

Vehicles A and B approach each other with an angle α between their lines of motion (directions of their velocity vectors) as shown in Figure 1.

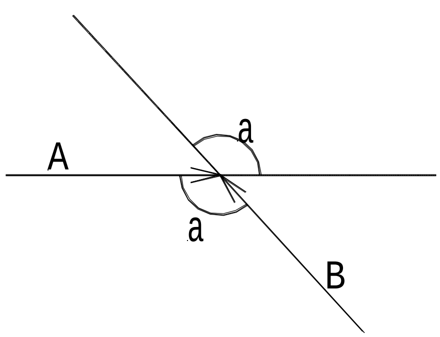

Figure 1. Directions of travel of vehicles A and B at impact.

Vehicle A has a velocity relative to the earth of $\vec{V}_{AE}$ and B has a velocity relative to the earth of $\vec{V}_{BE}$. The earth's velocity relative to B is thus $\vec{V}_{EB} = -\vec{V}_{BE}$.

The velocity of A relative to B (A's velocity of approach as seen by an observer riding in B) is $\vec{V}_{AB}$. This is the vector difference $\vec{V}_{AE} - \vec{V}_{BE}$.

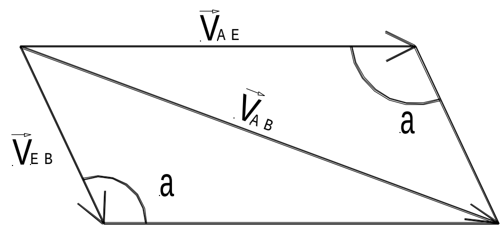

Figure 2. Closing speed from the parallelogram method of vector addition.

Thus, by vector addition we get the closing speed V_{AB} i.e., $\vec{V}_{AB} = \vec{V}_{AE} + \vec{V}_{EB} = \vec{V}_{AE} - \vec{V}_{BE}$. This is shown in the diagram of vector addition by the parallelogram method in Figure 2. Also note an observer in A sees B approaching at velocity $\vec{V}_{BA}$ which is just $-\vec{V}_{AB}$.

Applying the Cosine Law to either of the triangles in Figure 2 we have:

$$V_{AB}^2 = V_{AE}^2 + V_{EB}^2 - 2V_{AE}V_{EB}Cos\alpha \,. \tag{1}$$

Now consider the Law of Conservation of Momentum. By the definition of a completely inelastic collision, the two colliding objects have the same final velocity (same final speed and same final direction). The initial momentum lines of action will be the same as the lines of action of the initial velocities, but will be of different magnitude (length in diagram) since momentum = $M\vec{V}$.

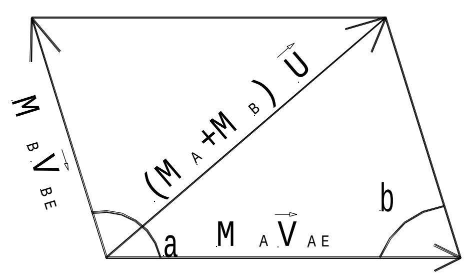

Figure 3. Momentum vectors of each vehicle and total momentum.

Again from the Cosine Law and the Law of Conservation of Momentum as shown in Figure 3:

$$\{(M_A + M_B)\cdot U\}^2 = (M_A V_{AE})^2 + (M_B V_{BE})^2 - 2(M_A V_{AE} \cdot M_B V_{BE})\cdot Cos\beta \tag{2}$$

where $\vec{U}$ is the post-impact velocity of both vehicles.

Note that $\angle\beta$ is the supplement of $\angle\alpha$ thus

$$Cos\beta = -Cos\alpha \, . \tag{3}$$

We can rewrite equation (2) as:

$$(M_A + M_B)^2 U^2 - M_A^2 V_{AE}^2 - M_B^2 V_{BE}^2 = 2(M_A V_{AE} \cdot M_B V_{BE}) \cdot Cos\alpha \, . \tag{4}$$

From equation (1) multiplied by $M_A M_B$:

$$2M_A V_{AE} M_B V_{BE} Cos\alpha = M_A M_B V_{AE}^2 + M_A M_B V_{BE}^2 - M_A M_B V_{AB}^2 \, . \tag{5}$$

Eliminating the terms containing $Cos\alpha$ from equations (4) and (5), i.e. setting the left side of equation (4) equal to the right side of equation (5), we have:

$$(M_A + M_B)^2 U^2 - M_A^2 V_{AE}^2 - M_B^2 V_{BE}^2 = M_A M_B V_{AE}^2 + M_A M_B V_{BE}^2 - M_A M_B V_{AB}^2 \, . \tag{6}$$

The approach speed, or closing speed, at impact is the magnitude of the vector $\vec{V}_{AB}$ (or $|\vec{V}_{AB}|$, or the scalar, V_{AB}).

Divide equation (6) by $M_A M_B$ and solve for V_{AB}^2:

$$V_{AB}^2 = V_{AE}^2 + V_{BE}^2 - \frac{(M_A + M_B)^2 U^2}{M_A M_B} + \frac{M_A V_{AE}^2}{M_B} + \frac{M_B V_{BE}^2}{M_A} \, . \tag{7}$$

From the Law of Conservation of Energy, initial kinetic energy equals final kinetic energy plus energy lost in crushing the vehicles, E (note $E = E_A + E_B$).

$$\frac{1}{2} M_A V_{AE}^2 + \frac{1}{2} M_B V_{BE}^2 = \frac{1}{2}(M_A + M_B) \cdot U^2 + E \, . \tag{8}$$

From equation (8) we get:

$$U^2 = \frac{M_A V_{AE}^2 + M_B V_{BE}^2 - 2E}{(M_A + M_B)} \, . \tag{9}$$

Substituting for U^2 from (9) into (7):

$$V_{AB}^2 = V_{AE}^2 + V_{BE}^2 - (M_A + M_B)\left[\frac{M_A V_{AE}^2 + M_B V_{BE}^2 - 2E}{M_A M_B}\right] + \frac{M_A V_{AE}^2}{M_B} + \frac{M_B V_{BE}^2}{M_A} \, . \tag{10}$$

Expanding the bracketed terms in (10):

$$V_{AB}^2 = V_{AE}^2 + V_{BE}^2 - \frac{M_A V_{AE}^2}{M_B} - V_{AE}^2 - V_{BE}^2 - \frac{M_B V_{BE}^2}{M_A} + \frac{(M_A + M_B)2E}{M_A M_B} + \frac{M_A V_{AE}^2}{M_B} + \frac{M_B V_{BE}^2}{M_A} \, . \tag{11}$$

Simplifying and solving for V_{AB} we have:

$$V_{AB} = \sqrt{\frac{M_A + M_B}{M_A M_B} 2E} \, . \tag{12}$$

Substituting $M = \frac{W}{g}$ where W is the weight and g the gravitational acceleration:

$$V_{AB} = \sqrt{\frac{W_A + W_B}{W_A W_B} 2gE} \, . \tag{13}$$

This result is identical to the results derived for the special case of in-line completely inelastic collisions (e.g. see [1] p. 90-31, equation 109, and [2], p. 206, equation 7-1).

GENERAL CASE

As we have just seen in eliminating the post impact speed, U, the $Cos\alpha$ term also dropped out, thus the general completely inelastic collision case relation is identical to that of the collinear case. This would be the expected result if we consider the collision from a center of mass coordinate system (see e.g. [3] and [4]). This coordinate system is frequently used in calculations of nuclear reaction kinematics.

The derivation of equation (13) was carried out in the "laboratory coordinate system" whose origin is fixed relative to the earth's surface. If we transform the collision to a coordinate system whose origin is the center of mass (c.m.) of the systems, the motion is reduced from motion of the two colliding objects in a plane, to motion of the objects along a line toward or away from the center of mass of the system. In the c.m. system, the sum of the momenta of the two colliding objects (the momentum of the system) is always zero, and the coordinates are sometimes called zero momentum coordinates.

This motion of the c.m. is easily determined by solving for the motion of the two objects after a completely inelastic collision, and projecting it backward to times before collision, if needed. In this system, we have one-dimensional motion with both objects moving along a straight line approaching the origin (the c.m.) before collision, and both objects receding away from the origin along a

straight line after collision (except for the case of a completely inelastic collision in which case both objects remain at the origin following collision). Except in the case of a direct head-on collision with exact rearward rebound, the line along which the two objects move before impact and on which the c.m. lies, and the line along which the objects move after impact do not coincide, but cross at the point of impact. We can see that the magnitude of the vector difference of the before-collision velocities in the laboratory coordinate system is just the speed at which the two objects approach each other along a straight line joining them and the c.m.; this is exactly the center of mass approach speed.

The collision described above in laboratory coordinates could be transformed into a c.m. coordinate system and a one-line motion derivation similar to the derivation in [1] and [5], appendix B, would result in the same results of equation (13) (see Appendix). This is consistent with the principle of classical relativity, which tells us that changing the frame of reference from which the event is described does not change the physics of the event.

Now consider a general collision, that is one somewhere from completely inelastic ($e = 0$), up to, but not including, perfectly elastic ($0 \le e < 1$) and for any incident angle ($0 \le \alpha \le 180^0$). Relying on classical relativity, let us describe the collision in the c.m. system and recognize the results as valid in the laboratory system in which we observe collisions.

For sign convention, let object A be to the left of the c. m., moving to the right, the positive direction, with speed (one dimensional velocity) relative to the c. m. of V_A before the collision and with speed U_A away from the c.m. after collision. Object B is initially to the right of the c. m. moving left toward the c.m. with a speed V_B before collision and with speed U_B away from the c. m. after collision (except if $e = 0$, where $U_A = U_B = 0$).

The coefficient of restitution mentioned earlier is defined as the ratio of the separation speed to the approach speed (multiplied by -1 so e will always be positive or zero).

$$e = -\frac{U_A - U_B}{V_A - V_B} = \frac{U_B - U_A}{V_A - V_B}. \tag{14}$$

(Note that in the center of mass coordinate system described above $V_A > 0,\ V_B < 0\quad U_A \le 0\quad U_B \ge 0$.

The Law of Conservation of Momentum for this collision:

$$M_A V_A + M_B V_B = M_A U_A + M_B U_B\ , \tag{15}$$

the Law of Conservation of Energy for this collision:

$$\frac{1}{2}M_A V_A^2 + \frac{1}{2}M_B V_B^2 = \frac{1}{2}M_A U_A^2 + \frac{1}{2}M_B U_B^2 + E\ , \tag{16}$$

and the total kinetic energy dissipated, $E = E_A + E_B$, is the sum of that dissipated in damage to each vehicle.

To get an equation similar in form to equations (12) and (13), we must eliminate U_A and U_B from equations (14), (15) and (16), and solve for the closing speed at impact, $V_{CL} = V_A - V_B$, as a function of vehicle weights, g, e and E. From equation (14):

$$U_B = eV_A - eV_B + U_A\ . \tag{17}$$

Substitute U_B from equation (17) into equation (15):

$$M_A V_A + M_B V_B = M_A U_A + M_B\left(eV_A - eV_B + U_A\right)\ . \tag{18}$$

Rearranging equation (18):

$$M_A V_A + M_B V_B = \left(M_A + M_B\right)\cdot U_A + M_B\left(eV_A - eV_B\right)\ . \tag{19}$$

or solving for U_A,

$$\frac{M_A V_A + M_B\left[V_B - \left(eV_A - eV_B\right)\right]}{\left(M_A + M_B\right)} = U_A\ . \tag{20}$$

Squaring both sides of equation (20):

$$\frac{M_A^2V_A^2 + 2M_AM_BV_A\left[V_B - \left(eV_A - eV_B\right)\right] + M_B^2\left[V_B - \left(eV_A - eV_B\right)\right]^2}{\left(M_A + M_B\right)^2} = U_A^2 \ . \tag{21}$$

Now consider the conservation of energy equation (16) and substitute U_B from equation (17) into equation (16):

$$M_AV_A^2 + M_BV_B^2 = M_AU_A^2 + M_B\left(eV_A - eV_B + U_A\right)^2 + 2E \ . \tag{22}$$

Expand equation (22) and rearrange terms:

$$M_AV_A^2 + M_BV_B^2 = M_AU_A^2 + M_B\left(eV_A - eV_B\right)^2 + M_BU_A^2 + 2M_B\left(eV_A - eV_B\right)\cdot U_A + 2E \ . \tag{23}$$

Simplifying:

$$M_AV_A^2 + M_BV_B^2 = \left(M_A + M_B\right)\cdot U_A^2 + M_B\left(eV_A - eV_B\right)^2 + 2M_B\left(eV_A - eV_B\right)\cdot U_A + 2E \ . \tag{24}$$

Solve for U_A^2:

$$\frac{M_AV_A^2 + M_BV_B^2 - M_B\left(eV_A - eV_B\right)^2 - 2M_B\left(eV_A - eV_B\right)\cdot U_A - 2E}{\left(M_A + M_B\right)} = U_A^2 \ . \tag{25}$$

Since the left sides of both equations (21) and (25) are equal to U_A^2, set them equal:

$$\frac{M_AV_A^2 + M_BV_B^2 - M_B\left(eV_A - eV_B\right)^2 - 2M_B^2\left(eV_A - eV_B\right)\cdot U_A - 2E}{\left(M_A + M_B\right)} =$$

$$\frac{M_A^2V_A^2 + 2M_AM_BV_A\left[V_B - \left(eV_A - eV_B\right)\right] + M_B^2\left[V_B - \left(eV_A - eV_B\right)\right]^2}{\left(M_A + M_B\right)^2} \ . \tag{26}$$

Substitute for U_A, from equation (20) into equation (26):

$$\frac{M_AV_A^2 + M_BV_B^2 - M_B\left(eV_A - eV_B\right)^2 - 2M_B\left(eV_A - eV_B\right)\left\{\dfrac{M_AV_A + M_B\left[V_B - \left(eV_A - eV_B\right)\right]}{\left(M_A + M_B\right)}\right\} - 2E}{\left(M_A + M_B\right)} =$$

$$\frac{M_A^2V_A^2 + 2M_AM_BV_A\left[V_B - \left(eV_A - eV_B\right)\right] + M_B^2\left[V_B - \left(eV_A - eV_B\right)\right]^2}{\left(M_A + M_B\right)^2} \ . \tag{27}$$

Multiply equation (27) by $\left(M_A + M_B\right)$ and expand the last term of the left side:

$$M_AV_A^2 + M_BV_B^2 - M_B\left(eV_A - eV_B\right)^2 + \left\{\frac{-2M_AM_BV_A\left(eV_A - eV_B\right) - 2M_B^2\left(eV_A - eV_B\right)\left[V_B - \left(eV_A - eV_B\right)\right]}{\left(M_A + M_B\right)}\right\} - 2E =$$

$$\frac{M_A^2V_A^2 + 2M_AM_BV_A\left[V_B - \left(eV_A - eV_B\right)\right] + M_B^2\left[V_B - \left(eV_A - eV_B\right)\right]^2}{\left(M_A + M_B\right)} \ . \tag{28}$$

Extract $2E$ and multiply both sides of the equation by $\left(M_A + M_B\right)$ again:

$$2E\left(M_A + M_B\right) = \left(M_A + M_B\right)\left\{M_AV_A^2 + M_BV_B^2 - M_B\left(eV_A - eV_B\right)^2\right\} - 2M_AM_BV_A\left(eV_A - eV_B\right) -$$

$$2M_B^2\left(eV_A - eV_B\right)\left[V_B - \left(eV_A - eV_B\right)\right] - \left\{2M_AM_BV_A\left[V_B - \left(eV_A - eV_B\right)\right] + M_A^2V_A^2 + M_B^2\left[V_B - \left(eV_A - eV_B\right)\right]^2\right\} . \tag{29}$$

Continue expanding terms, and for brevity, let $(eV_1 - eV_2) = (\ )$ in equations (30) and (31):

$$2E(M_A + M_B) = M_A^2V_A^2 + M_AM_BV_B^2 - M_AM_B(\ )^2 + M_AM_BV_A^2 + M_B^2V_B^2 - M_B^2(\ )^2 - 2M_AM_BV_A(\ ) -$$

$$2M_B^2(\ )\cdot[V_B - (\ )] - M_B^2[V_B - (\ )]^2 - 2M_AM_BV_A[V_B - (\ )] - M_A^2V_A^2 \ . \qquad (30)$$

and

$$2E(M_A + M_B) = \overbrace{M_A^2V_A^2}^{A} + M_AM_BV_B^2 - M_AM_B(\ )^2 + M_AM_BV_A^2 + \overbrace{M_B^2V_B^2}^{B} - \overbrace{M_B^2(\ )^2}^{E} - \overbrace{2M_AM_BV_A(\ )}^{D} -$$

$$\underbrace{2M_B^2V_B(\ )}_{C} + \underbrace{2M_B^2(\ )^2}_{E} - \underbrace{M_A^2V_A^2}_{A} - \underbrace{M_B^2V_B^2}_{B} + \underbrace{2M_B^2V_B(\ )}_{C} - \underbrace{M_B^2(\ )^2}_{E} - 2M_AM_BV_AV_B + \underbrace{2M_AM_BV_A(\ )}_{D} \ . \qquad (31)$$

The pairs of terms marked A, B, C and D and the three terms marked E all add to zero.

Simplifying and dividing both sides of equation (31) by M_AM_B:

$$\frac{2E(M_A + M_B)}{M_AM_B} =$$

$$V_B^2 - (eV_A - eV_B)^2 + V_A^2 - 2V_AV_B \ . \qquad (32)$$

Rearranging terms and recall that $V_{CL} = V_A - V_B$:

$$\frac{2E(M_A + M_B)}{M_AM_B} = V_{CL}^2\left(1 - e^2\right) . \qquad (33)$$

Therefore:

$$V_{CL} = \sqrt{\frac{2E(M_A + M_B)}{M_AM_B\left(1 - e^2\right)}} \ , \qquad (34)$$

or,

$$V_{CL} = \sqrt{\frac{2gE(W_A + W_B)}{W_AW_B\left(1 - e^2\right)}} \ . \qquad (35)$$

CONCLUSIONS

We have now verified the equation for determining closing speed at impact from dissipated energy for the general case by deriving it from basic laws of physics. Closing speed is useful not only for the special cases described in the Introduction, but also in many other situations. In many cases, post-impact momentum can be determined, and with equation (35) giving a numerical value for $V_{CL} = V_A - V_B$, V_A or V_B can be eliminated from equation (15); then solutions for both V_A and V_B in the c. m. coordinates can be obtained and from that the earth frame of reference speeds can be determined if their pre-impact directions are known.

For high-speed non-glancing collisions, experience tells us $e \approx 0$, even though the vehicles do not always come to rest in contact with each other. For low-speed impacts where $e \neq 0$, low-speed crash test data can be a guide to a reasonable range of values of e or perhaps can establish an upper limit for e. Since closing speed depends on e as $\left(1 - e^2\right)^{-1/2}$, calculated closing speed for small values of e yield very small differences from closing speed for completely inelastic collisions. For example, $e = 0.5$ yields a speed only 15% higher than for $e = 0$, however, the difference rises rapidly for $e > 0.5$ (it is approximately 50% higher for $e = 0.75$).

Note that in equation (34) and (35) when $e \rightarrow 0$ (completely inelastic collision), equation (35) becomes equation (13). For a perfectly elastic collision $e \rightarrow 1$ and the denominator of (35) becomes zero. However, for a perfectly elastic collision $E \equiv 0$ by definition, and $E/\left(1 - e^2\right)$ of equation (35) is indeterminate and thus cannot describe a perfectly elastic collision.

In both derivations above, it is assumed that the post-impact rotational kinetic energy is negligible and that energy dissipated by sliding of vehicle surfaces during contact is negligible.

In reference [5], Bruno Schmidt et al., very nicely describe and compare actual experimental collisions with (a) both vehicles moving, and (b) with the target vehicle at rest and with closing speeds approximately equal for the two tests. Also in appendix C, he derives the dissipated energy in terms of weights, closing speed, and coefficient of restitution. This result solved for closing speed is identical to equation (35).

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APPENDIX : COMPLETELY INELASTIC COLLISION IN CENTER OF MASS COORDINATES:

Conservation of momentum in c.m. coordinate system:

$$A - - - - - > \underset{c.m.}{\bullet} < - - - B$$

$$M_A V_A + M_B V_B = 0 . \quad \text{(A1)}$$

Conservation of energy for a completely inelastic collision:

$$\frac{1}{2} M_A V_A^2 + \frac{1}{2} M_B V_B^2 - E = 0 . \quad \text{(A2)}$$

Closing speed: $V_{CL} = V_A - V_B$. (A3)

Square of equation (A1):

$$M_A^2 V_A^2 + M_B^2 V_B^2 + 2 M_A M_B V_A V_B = 0 . \quad \text{(A4)}$$

Square of equation (A3): $V_A^2 + V_B^2 - V_{CL}^2 = 2 V_A V_B$. (A5)

Substitute (A5) into (A4), divide by $M_A M_B$, and collect terms:

$$V_A^2 \left(\frac{M_A}{M_B} + 1 \right) + V_B^2 \left(\frac{M_B}{M_A} + 1 \right) = V_{CL}^2 . \quad \text{(A6)}$$

$$\left(M_A + M_B \right) \left(\frac{V_A^2}{M_B} + \frac{V_B^2}{M_A} \right) = V_{CL}^2 . \quad \text{(A7)}$$

$$\frac{M_A + M_B}{M_A M_B} \left(M_A V_A^2 + M_B V_B^2 \right) = V_{CL}^2 . \quad \text{(A8)}$$

Substitute E from (A2), solve for

$$V_{CL} : V_{CL} = \sqrt{\frac{\left(M_A + M_B \right) 2E}{M_A M_B}} = \sqrt{\frac{\left(W_A + W_B \right) 2E}{W_A W_B}} . \quad \text{(A9)}$$